Quantization of a one-dimensional time-dependent periodic system with Hamiltonian and constants of motion approaches

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ABSTRACT

For a particle moving in a one-dimensional space an under a periodic external force, its quantization is study using the Hamiltonian (generalized linear momentum quantization) and constant of motion (velocity quantization) approaches. it is shown a great difference on the quantization of both approaches and the ambiguities arisen by using the quantization on the constants of motion.

1. Introduction

Notwithstanding the great success of the Hamiltonian and Lagrangian [1] approaches to quantize a classical dynamical system [2], there are still some dynamical systems which their consistent quantization can be questioned [3,4]. In particular, the so called dissipative systems [5] and time-explicitly depending systems [6] present some problems for their consistent formulation. One of the main problems that a one-dimensional dissipative system presents is that most of the times it is not possible to have explicitly the velocity in terms of the generalized linear momentum and position, $v = v(x, p)$, from the usual definition of the generalized linear momentum $p = \partial L / \partial v$, where L is the associated Lagrangian of the system. In turns, this means that the Hamiltonian associated to this system, $H = vp - L(x, v)$, can not be given explicitly and remains implicit within the associated constant of motion [7], $K = K(x, v)$. It is shown [8] that a consistent quantization for this type of systems can be gotten by using the association of an Hermitian operator to the velocity variable, $\hat{v} = -i(\hbar/m)\partial/\partial x$, the constant of motion, $\hat{K}(x, \hat{v})$, and proceeding to solve the associated Shrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{K}(x, \hat{v})\Psi \ , \tag{1}$$

where $\Psi = \Psi(x, t)$ is the wave function. The question arises whether or not this same approach can be extended to nonautonomous systems (explicitly time dependent systems). In this paper, a one-dimensional time-dependent periodic dynamical system is studied to see whether or not a natural consistent extension of Eq. (1) can be achieved for this system. In the first part, the Hamiltonian quantization approach is used and its known natural ambiguity is pointed out. In the second part, three classical constants of motion are deduced with units of energy and their quantization is carried out through the extension of Eq. (1). Finally, the ambiguity resulting from this quantization approach is pointed out.

2. Hamiltonian quantization approach

Consider the following nonautonomous dynamical system

$$\dot{x} = p/m \ , \qquad \dot{p} = -A \cos \omega t \ , \tag{2}$$

where p is the generalized linear momentum, x represents the position of the particle, m is the mass of the particle, ω and A are the angular frequency and amplitude of the external oscillating force. A Hamiltonian associated to this system is given by

$$H = \frac{p^2}{2m} = xA \cos \omega t \ . \tag{3a}$$

Note that one gets the following limits

$$\lim_{A \to 0} H = \frac{p^2}{2m} \qquad \text{and} \qquad \lim_{\omega \to 0} H = \frac{p^2}{2m} + Ax \tag{3b}$$

which are what one could expect with the corresponding limits in Eq. (2). Furthermore, one could add an arbitrary time dependent function to Eq. (3a) and

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still having a Hamiltonian which the Hamilton equations satisfy Eq. (2). In fact, this is true for any nonautonomous dynamical system. Then, one can say that two Hamiltonian H and H' are equivalent if they are related by the following expression

$$H' = H + f(t) \ , \tag{4}$$

where $f(t)$ is an arbitrary function. This defines an equivalence relation [10] where a nonautonomous dynamical system is characterized by a class of Hamiltonians

$$[H] = \left\{ H'(Hamiltonian)| \ H' = H + f(t) \right\} \ . \tag{5}$$

This ambiguity, however, does not affect the classical dynamics neither the quantum dynamics, as one will sees below.

To quantize our system, one needs to solve the associated Shrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}(x, \hat{p}) \Psi \ , \tag{6}$$

where $\Psi = \Psi(x, t)$ is the wave function, $\widehat{H}$ and $\hat{p} = i\hbar \partial / \partial x$ are the Hermitian operators associated to the classical variables H and p. Consider also the expansion of the wave function Ψ in terms of the basis $\{|k\rangle\}_{k \in \Re}$,

$$\Psi(x, t) = \int dk \ C(k, t) |k\rangle \ , \tag{7}$$

where k is given by $k = \sqrt{2mE_k/\hbar^2}$, being E_k the energy of the free particle. The basis $\{|k\rangle\}_{k \in \Re}$ has the following relations

$$\langle x|k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} \ , \qquad \langle k'|k\rangle = \delta(k - k') \ , \qquad \hat{p}|k\rangle = \hbar k|k\rangle \tag{8a}$$

with the braket $\langle k'|k\rangle$ defined as

$$\langle k'|k\rangle = \int dx \ \langle k'|x\rangle\langle x|k\rangle \ . \tag{8b}$$

Using (8b) and (3a), substituting (7) in (6), and multiplying to the left by the bra $\langle k'|$, one gets the a first order partial differential equation for C

$$i\hbar C(k', t) = E_{k'} C(k', t) + iA \cos \omega t \frac{\partial C(k', t)}{\partial k'} \ . \tag{9}$$

One can simplify this equation doing the following change of variable

$$C(k, t) = e^{-iE_k t/\hbar} D(k, t) \ . \tag{10}$$

The resulting equation for the $D's$ coefficients is

$$\frac{\partial D}{\partial t} - \frac{A}{\hbar} \cos \omega t \frac{\partial D}{\partial k} = 0 \tag{11}$$

which can be solved by the characteristics method [9]. Given the initial condition $C(k, 0) = D(k, 0) = F(k)$, one has

$$D(k, t) = F\left(k + \frac{A}{\hbar\omega} \sin \omega t\right) \ . \tag{12}$$

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Of course, the initial condition $C(k, 0) = F(k)$ is related with the initial wave function $\Psi_0(x) = \Psi(x, 0)$ as $\Psi_o(x) = \int dk\, F(k)|k\rangle$. That is, $F(k)$ is just the Fourier transformation of $\Psi_0(x)$. Thus, the coefficients $C's$ are given by

$$C(k, t) = F\left(k + \frac{A}{\hbar\omega}\sin\omega t\right) e^{-i\hbar k^2 t/2m} . \qquad (13)$$

With this expression, the solution of Eq. (6) is fully determined. Now, the ambiguity (4) will be reflected in Eq. (13) by an additional phase, $-(i/\hbar)\int f(t)dt$ which does not depend on the variable k, and this arbitrary phase corresponds to an arbitrary local element of the unitary group $U(1)$. However, since $|\Psi(x, t)|^2$ gives the probability to find the particle in the position x at the time t, this global phase has not contribution on the quantum dynamics of the system.

3. Constants of motion quantization approach

The nonautonomous dynamical system (2) can be written in terms of the position, x, and velocity v, variables as

$$\dot{x} = v , \qquad \dot{v} = -\frac{A}{m}\cos\omega t , \qquad (14)$$

and a constant of motion for this system is a function $K = K(x, v, t)$ such that $dK/dt = 0$. That is, it satisfies the following first order partial differential equation

$$v\frac{\partial K}{\partial x} - \frac{A}{m}\cos\omega t\frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0 . \qquad (15)$$

This equation can be solved by the characteristics method [9], bringing about the following characteristics curves

$$C_1 = v + \frac{A}{m\omega}\sin\omega t \qquad (16a)$$

and

$$C_2 = t\left(v + \frac{A}{m\omega}\sin\omega t\right) + \frac{A}{m\omega^2}(\cos\omega t - 1) - x . \qquad (16b)$$

Therefore, the general solution of Eq. (15) is given by

$$K = G(C_1, C_2) , \qquad (17)$$

where G is an arbitrary function of both characteristics curves. Let us note that C_1 and C_2 have the following limits

$$\lim_{\omega\to 0} C_1 = v + \frac{At}{m} , \qquad \lim_{\omega\to 0} C_2 = t\left(v + \frac{At}{m}\right) - \frac{At^2}{2m} - x \qquad (18a)$$

and

$$\lim_{A\to 0} C_1 = v , \qquad \lim_{A\to 0} C_2 = vt - x . \qquad (18b)$$

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Now, there are at least three different ways to get constants of motion with units of energy. One could select the function G as $G = (m/2)C_1^2$ to get the constant of motion

$$K_1(x, v, t) = \frac{1}{2}mv^2 + \frac{vA}{\omega}\sin\omega t + \frac{A}{2m\omega^2}\sin^2\omega t \ . \tag{19}$$

Other way of selecting G is of the form $G = (m/2)C_1^2 - AC_2$. This selection brings about the following constant of motion

$$\begin{aligned}
K_2(x, v, t) &= \frac{m}{2}\left(v^2 + \frac{2vA}{m\omega}\sin\omega t + \frac{A^2}{m^2\omega^2}\sin^2\omega t\right) \\
&\quad - Atv - \frac{A^2t}{m\omega}\sin\omega t - \frac{A^2}{m\omega^2}(cos\omega t - 1) + Ax \ .
\end{aligned} \tag{20}$$

In addition, one could select G of the form $G = (m\omega/2)C_1C_2$ to get the constant of motion

$$\begin{aligned}
K_3(x, v, t) &= \frac{m\omega t}{2}\left(v^2 + \frac{2vA}{m\omega}\sin\omega t + \frac{A^2}{m^2\omega^2}\sin^2\omega t\right) + \frac{Av}{m\omega}(\cos\omega t - 1) \\
&\quad + \frac{A^2}{2m\omega^2}\sin\omega t(cos\omega t - 1) - \frac{m\omega}{2}\left(xv + \frac{Ax}{m\omega}\sin\omega t\right) \ .
\end{aligned} \tag{21}$$

These constants of motion have units of energy and have the following limits

$$\lim_{\omega\to 0} K_1 = \frac{m}{2}\left(v - \frac{At}{m}\right)^2 \ , \qquad \lim_{A\to 0} K_1 = \frac{1}{2}mv^2 \ , \tag{22a}$$

$$\lim_{\omega\to 0} K_2 = \frac{1}{2}mv^2 + Ax, \qquad \lim_{A\to 0} K_2 = \frac{1}{2}mv^2 \ , \tag{22b}$$

and

$$\lim_{\omega\to 0} K_3 = 0, \qquad \lim_{A\to 0} K_3 = 0 \ . \tag{22c}$$

The quantization of the system (14) through the constants of motion will be carried out with the association of an Hermitian operator to the velocity, $\hat{v} = -(i\hbar/m)\partial/\partial x$, and the constant of motion, $\widehat{K}(x, \hat{v})$. Then, one will proceed to solve the Shrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{K}(x, \hat{v}, t)\Psi \ . \tag{23a}$$

Note that within this velocity quantization approach, the Heisenberg's uncertainty relation is expressed as

$$\Delta x \Delta v \geq \frac{\hbar}{m} \ , \tag{23b}$$

and the following relations are gotten straightforwardly

$$[x, x] = [\hat{v}, \hat{v}] = 0 \qquad [x, \hat{v}] = \frac{i\hbar}{m} \ , \quad \text{and} \quad \hat{v}|k\rangle = \frac{\hbar k}{m}|k\rangle \ , \tag{24}$$

where $[,]$ represents the commutator of two operators, and the state $|k\rangle$ is given by (8a). Now, proposing the wave function of the form (7), multiplying to the left by

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the bra $|k'\rangle$, and using the above properties, one gets for the above three constants of motion the following equations associated to their coefficients (renaming k' by k)

$$i\hbar \dot{C}^{(1)}(k,t) = \left[\frac{\hbar^2 k^2}{2m} + \frac{A\hbar k}{m\omega}\sin\omega t + \frac{A}{2m\omega^2}\sin^2\omega t\right] C^{(1)}(k,t) , \qquad (25a)$$

$$i\hbar \dot{C}^{(2)}(k,t) = B(k,t) C^{(2)}(k,t) + iA\frac{\partial C^{(2)}(k,t)}{\partial k} , \qquad (25b)$$

and

$$i\hbar \dot{C}^{(3)}(k,t) = f(k,t) C^{(3)}(k,t) + \frac{i\hbar\omega}{4} C^{(3)}(k,t) - \frac{im\omega}{2}\left[\frac{\hbar k}{m} + \frac{A}{m\omega}\sin\omega t\right]\frac{\partial C^{(3)}(k,t)}{\partial k} , \qquad (25c)$$

where the functions $B(k,t)$ and $f(k,t)$ have been defined as

$$\begin{aligned}
B(k,t) &= \frac{\hbar^2 k^2}{2m} + \frac{A\hbar k}{m\omega}\sin\omega t + \frac{A}{2m\omega^2}\sin^2\omega t \\
&\quad - \frac{A\hbar kt}{m} - \frac{A^2 t}{m\omega}\sin\omega t + \frac{A^2}{\omega^2}(1 - \cos\omega t)
\end{aligned}$$

$$(26a)$$

and

$$\begin{aligned}
f(k,t) &= \frac{m\omega t}{2}\left[\frac{\hbar^2 k^2}{m^2} + \frac{2A\hbar k}{m^2\omega}\sin\omega t + \frac{A^2}{m^2\omega^2}\sin^2\omega t\right] \\
&\quad + \frac{A\hbar k}{2m\omega}(\cos\omega t - 1) + \frac{A^2}{2m\omega^2}\sin\omega t(\cos\omega t - 1) .
\end{aligned}$$

$$(26b)$$

Note that in order to associate an Hermitian operator to the constant of motion K_3, one had to make use of the Weyl's quantization to the product xv, $\widehat{xv} = (x\hat{v} + \hat{v}x)/2$. Eqs. (25's) represent first order partial differential equations, and given the initial condition $C^{(i)}(k,0) = F(k)$ for $i = 1, 2, 3$, their solutions are given by

$$C^{(1)}(k,t) = F(k)e^{-i\phi_1(k,t)} , \qquad (27a)$$

$$C^{(2)}(k,t) = F\left(k + \frac{At}{\hbar}\right)e^{-i\phi_2(k,t)} , \qquad (27b)$$

and

$$C^{(3)}(k,t) = F\left(ke^{-\omega t/2} + g(t) - g(0)\right)e^{\omega t/4 - i\phi_3(k,t)} , \qquad (27c)$$

where the phases ϕ_1, ϕ_2, and ϕ_3 and the function $g(t)$ have been defined as

$$\phi_1(k,t) = \frac{\hbar k^2 t}{2m} - \frac{kA}{m\omega^2}(\cos\omega t - 1) + \frac{A}{2m\omega^2}\left(\frac{t^2}{2} - \frac{\sin 2\omega t}{4\omega}\right) , \qquad (28a)$$

$$\phi_2(k,t) = \frac{1}{\hbar}\int_0^t B\left(k + \frac{At}{\hbar} - \frac{As}{\hbar}, s\right) ds , \qquad (28b)$$

$$\phi_3(k,t) = \frac{1}{\hbar}\int_0^t f\left(e^{\omega s/2}(k - e^{-\omega t/2} + g(t) - g(s)), s\right) ds , \qquad (28c)$$

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and

$$g(t) = \frac{A}{5\hbar\omega}(2\cos\omega t + A\sin\omega t) \ . \qquad (28d)$$

Note from Eq. (27c) that the real exponent has to be canceled with the negative exponent coming from the initial conditions to keep the probability finite at any time. On the other hand, the probabilities $|C^{(i)}(k,t)|^2$ for i=1,2,3 tell us quite different behavior of the quantum system. The solution (27a) tell us essentially that the quantum system described by the constant of motion (19) will remain in the same initial quantum state but with a complicated phase. Solution (17b) tell us that the quantum system will change linearly its state of momentum k with respect the time. Finally, the solution (27c) tell us that the quantum system described by the constant of motion (21) will oscillate with respect the time in a form determined by (28d).

These ambiguities about selecting a proper constant of motion to quantize the dynamical nonautonomous system (14) make the approach of using (23a) a little bite subtle.

4. Conclusions and Comments

The study of Hamilton and constants of motion quantization approaches has been made for a one-dimensional nonautonomous periodic dynamical system. It was shown that both approaches brings about different solutions for the quantized system. In addition, it is not clear how to select a proper constant of motion to have the same solution as the Hamiltonian approach has, Eq. (13). At first sight, one might think that since the limits (3b) and (22b) are consistent, their associated quantum solutions (13) and (27b) should be also consistent (look alike). However, this is not the case. Therefore, a possible extension of Eq. (1) (autonomous systems) to Eq. (23a) (nonautonomous systems) is a little bite subtle.

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